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Quantum Logical Solution to the Measurement Problem of Quantum Mechanics

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In this paper I propose a reformulation and solution of the measurement problem of quantum mechanics. The reformulation depends on a quantum logical interpretation of quantum mechanics, broadly construed. The solution depends on a theorem about partial Boolean algebras which is proved here.

1. THE TRANSITION FROM CLASSICAL TO QUANTUM MECHANICS

A classical system is described in terms of a commutative algebra of dynamical quantities. These quantities are all real-valued functions of the state of the system and take values at all times, even when the system is interacting with other systems. The set of states is given by all possible assignments of values to a privileged set of quantities, the (generalized) positions and momenta of the system. So a classical state maps the set of dynamical quantities onto a set of corresponding values. Since these quantities $A, B, \ldots$ form a commutative algebra, the subalgebra of idempotent quantities (quantities satisfying the condition $A^2 = A$) is a Boolean algebra. A classical state maps the idempotent quantities—representing all possible properties of the system—onto the values 1 and 0, and this mapping is a 2-valued homomorphism on the algebra. So we can think of a classical state as assigning a value "yes" or "no" to every experiment designed to ascertain whether the system has a particular property or not, or we can think of the state as assigning a binary truth value, "true" or "false," to each classical proposition (asserting that the corresponding property is a property of the system in the given state), or we can think of the state as partitioning the set of possible properties of the system into the properties possessed by

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the system in the state and the properties not possessed by the system in the state.

A dynamical quantity is a magnitude associated with a set of possible values. An idempotent quantity is a quantity with only two possible values, 1 or 0. The algebra of dynamical quantities can be generated by the subalgebra of idempotent quantities, roughly because each quantity, e.g., position, corresponds to a set of idempotents, in this case the set of 2-valued quantities associated with each range of position. To assign a value to the position of a particle is equivalent to assigning a 1 to every range of positions containing the value and a 0 to every other range of positions. In other words, assigning a value to every classical quantity is equivalent to assigning a truth value to every classical proposition.

I shall refer to the algebra of idempotent dynamical quantities of a system as the "property structure" of the system. The property structure of a classical system is a Boolean algebra. It represents, through its ultrafilters, all possible ways in which the system can manifest its properties, or all possible ways in which the properties of the system can fit together as simultaneously determinate sets. To say that the (propositional) logic of classical mechanics is Boolean is to say that the class of models over which validity and associated semantic notions are defined, for the propositions of classical mechanics that assign ranges of values to dynamical quantities, is the class of Boolean property structures.

The transition from classical to quantum mechanics involves the transition from a commutative algebra of dynamical quantities to a noncommutative algebra, equivalently the transition from a Boolean to a non-Boolean algebra of idempotent quantities. This is, in effect, the formal significance of quantization. As in the classical case, the noncommutative algebra of dynamical quantities can be generated from the algebra of idempotents, the non-Boolean property structure of a quantum mechanical system. That the structure can be interpreted as a *property* structure in an analogous sense to the Boolean property structure of a classical system depends on the following analysis and the measurement theorem proved below.

The non-Boolean property structure of a quantum mechanical system can be represented as a partial Boolean algebra, in effect a family of Boolean algebras that are pasted together by identifying certain elements. Consider, for example, a quantum mechanical system associated with a 3-dimensional Hilbert space. Each set of three orthogonal 1-dimensional subspaces defines an 8-element Boolean algebra generated by these subspaces as atoms of the algebra. The algebra contains the three atoms, the three planes spanned by these atoms pairwise, together with the zero element (0) corresponding to the null subspace, and the unit element (1) corresponding to the whole 3-dimensional space. Evidently, some of these 8-element Boolean algebras have elements in common. For example, if we fix a 1-dimensional subspace $\mathscr{K}_1$ and consider two choices for the remaining pair of orthogonal lines—any initial pair $\mathscr{K}_2$, $\mathscr{K}_3$ orthogonal to $\mathscr{K}_1$, and any other pair $\mathscr{K}'_2$, $\mathscr{K}'_3$ orthogonal to $\mathscr{K}_1$—then the two 8-element Boolean algebras are pasted together at the elements 0, $\mathscr{K}_1$, $\mathscr{K}_1^\perp$, 1, where $\mathscr{K}_1^\perp$ represents the plane orthogonal to $\mathscr{K}_1$. The property structure of the system as a partial Boolean algebra is represented by the union of all the elements in each of the algebras generated by orthogonal triples of lines, together with the structural relations (order relations defined by subspace inclusion) holding between elements defined by the Boolean structure of the Boolean algebra to which the elements belong. I shall write $\mathscr{L} = \bigcup \mathscr{B}$ for this property structure, where the union is taken over all the (maximal) Boolean algebras generated by elements corresponding to orthogonal triples of lines in the Hilbert space $\mathscr{H}_3$. $\mathscr{L}$ contains maximal Boolean subalgebras (8-element subalgebras) and also nonmaximal Boolean subalgebras (4-element Boolean subalgebras generated by elements corresponding to a line and its orthogonal plane as atoms). Some of these are pasted together in $\mathscr{L}$ and some are not.

$\mathscr{L}$ represents the property structure of a quantum mechanical system. As constructed here, it is isomorphic to the partial Boolean algebra of subspaces of the Hilbert space of the system, or equivalently the partial Boolean algebra of projection operators of the system—just as the property structure of a classical mechanical system is a Boolean algebra $\mathscr{B}$ isomorphic to the Boolean algebra of (Borel) subsets of the phase space of the system, or equivalently the Boolean algebra of characteristic functions (0, 1 functions) on the Borel subsets of the phase space. Except in the special case of $\mathscr{H}_2$, $\mathscr{L}$ is not embeddable into any Boolean algebra. This means that there are no 2-valued homomorphisms on $\mathscr{L}$ in the general case. A 2-valued homomorphism on a classical property structure $\mathscr{B}$ is defined by each classical state and partitions the elements in $\mathscr{B}$ into those properties that belong to the system in the given state (elements assigned the value 1 by the homomorphism) and those properties that do not belong to the system in the state (elements assigned the value 0). So a 2-valued homomorphism is a classical truth value assignment on the propositions represented by the elements in $\mathscr{B}$, or a yes–no assignment on the corresponding properties represented by the elements in $\mathscr{B}$. As homomorphisms, these assignments respect the structure of $\mathscr{B}$.

We can express the difference between $\mathscr{L}$ and $\mathscr{B}$ this way: All the elements of $\mathscr{B}$ are *determinate* in each classical state, i.e., for every state, all the elements *have* a truth value (or yes–no value). What the truth value of a particular element is in a given state is determined or assigned by that state. This is not the case for a property structure $\mathscr{L}$ that is not embeddable into a Boolean algebra: only a proper subset of elements of $\mathscr{L}$ can be determinate for each quantum state. How are the determinate subsets defined? There are various possible proposals here. The most natural is the following "principle of determinateness:"

D: In a non-Boolean property structure $\mathscr{L}$ associated with a quantum mechanical system S, the elements that are determinate in the quantum state ψ are those elements e represented by projection operators P_e such that $P_\psi \leq P_e$ or $P_\psi \perp P_e$.

This makes all elements of $\mathscr{L}$ associated with subspaces *containing* the state or *orthogonal* to the state determinate. Other elements of $\mathscr{L}$ are indeterminate in the state ψ. So the elements determinate in the state ψ are those elements e assigned probability 1 ($P_\psi \leq P_e$) or 0 ($P_\psi \perp P_e$) by ψ.

There are two notions of state in classical mechanics: (1) the state s as a point in phase space, assigning values to all dynamical quantities, and (2) the state w as a probability measure on phase space. The first notion of state (call this a "property state") selects an ultrafilter of properties in the Boolean property structure $\mathscr{B}$, associated with propositions (assigning ranges of values to the dynamical quantities) that are true in s. This ultrafilter corresponds to the properties or propositions represented by subsets of phase space containing s—these are the properties possessed by the system in the state s. Properties represented by subsets of phase space not containing s are not properties of the system in the state s. Equivalently, s defines a 2-valued homomorphism on $\mathscr{B}$, with 1 corresponding to "true" or "possessed," and 0 corresponding to "false" or "not possessed." The second notion of state (call this a "statistical state") assigns probabilities to elements of $\mathscr{B}$.

In quantum mechanics, the quantum state ψ plays a dual role: it is both the analogue of the classical property state s and the analogue of the statistical state w. As the analogue of the classical statistical state w, ψ assigns probabilities to elements in $\mathscr{L}$. As the analogue of the classical property state s, ψ partitions $\mathscr{L}$ into elements that are true or possessed by the system in the state ψ, and elements that are false or not possessed by the system in the state ψ, *in a subset of elements of L that are determinate in the state* ψ (i.e., a subset of elements that are true or false in the state ψ, or possessed or not possessed by the system in the state ψ).

As defined by the principle D, this subset of elements of $\mathscr{L}$ generated by the state ψ has the following structure (in a finite-dimensional Hilbert space): ψ corresponds to an atom a_ψ in $\mathscr{L}$, the minimal, nonzero element in $\mathscr{L}$ representing the 1-dimensional subspace $\mathscr{H}_\psi$ or the projection operator P_ψ. Consider all the maximal Boolean subalgebras $\mathscr{B}_\psi$ in $\mathscr{L}$ that contain a_ψ as atom. The set $\mathscr{D}_\psi = \bigcup \mathscr{B}_\psi$ is the set of elements in $\mathscr{L}$ determinate in the state ψ, where the union is taken over all the (maximal) Boolean subalgebras containing a_ψ as atom. We can write $\mathscr{D}_\psi$ as $\mathscr{D}_\psi = \mathscr{D}_\psi^+ \cup \mathscr{D}_\psi^-$, where $\mathscr{D}_\psi^+ = \bigcup \mathscr{U}_\psi$, the union of the ultrafilters (maximal filters) generated by a_ψ

in all the $\mathscr{B}_\psi$, and $\mathscr{D}_\psi^- = \bigcup \mathscr{I}_\psi$, the union of the maximal ideals generated by the complement of a_ψ in all the B_ψ.

It is clear from the construction of $\mathscr{D}_\psi^+$ and $\mathscr{D}_\psi^-$ that the principle D selects determinate subsets of elements in $\mathscr{L}$, for each quantum state ψ, that are the appropriate non-Boolean or noncommutative analogues of the determinate subsets selected by a classical property state s (the subset $\mathscr{D}_s^+ = \mathscr{U}_s$ of properties possessed by a classical system in the state s, and the subset $\mathscr{D}_s^- = \mathscr{I}_s$ of properties that are not possessed by the system in the state s, where $\mathscr{D}_s^+ \cup \mathscr{D}_s^- = \mathscr{U}_s \cup \mathscr{I}_s = \mathscr{B}$). I have referred to the principle D as the "most natural" assumption here, but there are alternatives. Notably, Bohr's complementarity interpretation would not select the set of properties $\mathscr{D}_\psi^+$ as the set of properties possessed by a quantum mechanical system in the state ψ. The set $\bigcup \mathscr{B}_\psi$ is not a Boolean algebra (except in the case of $\mathscr{H}_2$, when $\bigcup \mathscr{B}_\psi$ reduces to a single maximal Boolean subalgebra in $\mathscr{L}$) and so $\mathscr{D}_\psi^+$ contains many incompatible (noncommuting) elements. For Bohr, the set of properties or propositions that are determinate for a quantum mechanical system at a particular time t (the set of dynamical quantities that can meaningfully be asserted to have values at t, or the set of concepts that are jointly applicable to the system at t) is not determined by the state ψ at t, but rather by the classically described measurement context (and so contains only mutually compatible elements).

2. THE MEASUREMENT PROBLEM REFORMULATED

How should we formulate the measurement problem for classical mechanics? In the case of optimal, ideal measurements, we want to show that, given an initial state for S and an initial state for the measuring instrument M (the zero indicator state), there exists an interaction that models the measurement of some S-quantity A by M, in the sense that the interaction correlates the values of A with the values of a physical quantity of M (the indicator values). This means that after a certain amount of time t, M should end up in some indicator state and S should end up in a state with a corresponding value of A, so that the value of the indicator quantity indicates the value of A. (Obvious modifications of this formulation for minimally disturbing measurements, or other realistic restrictions on measurements, do not concern the issues relevant here.) Since such interactions do, in principle, exist, the measurement problem, as a theoretical problem for classical mechanics, is (trivially) soluble.

How should the measurement problem be formulated for a theory of mechanics in which $\mathscr{B}$ is replaced by $\mathscr{L}$? What we want to show is that, given the initial quantum state ψ of a system S and the initial quantum state ρ_0 of a second system M (the measuring instrument) at some time t_0, where

ρ_0 represents an eigenvector of the zero value of the indicator quantity of M, there exists an interaction that transforms the state of the composite system $\psi \otimes \rho_0$ to a new state $\sigma \in \mathscr{H}_S \otimes \mathscr{H}_M$ at time t, in which (a) the properties of S associated with ranges of values of the measured quantity A are determinate in the state σ, (b) the properties of M associated with ranges of values of the indicator quantity of M are determinate in the state σ, and (c) the values of the measured quantity A are correlated with the indicator values of M in the state σ.

Now, it is well known that no such interaction exists. If $\psi = \alpha_k$, where α_k is an eigenstate of A corresponding to the eigenvalue a_k, only the transition $\psi \otimes \rho_0 \rightarrow \alpha_j \otimes \rho_j$ satisfies (a), (b), and (c), assuming the principle D, where ρ_j is an eigenvector of the jth eigenvalue r_j of the indicator quantity of M. (Adding a "no disturbance" requirement restricts j to k.) It follows that if $\psi = \sum c_i \alpha_i$, the interaction must yield the transition $\psi \otimes \rho_0 \rightarrow \sigma = \sum c_i \alpha_i \otimes \rho_i$, by linearity. Now requirement (c) is satisfied, but by the principle D, (a) and (b) both fail. The elements in $\mathscr{L}$ corresponding to the values a_i of A, i.e., the elements corresponding to the subspaces $\mathscr{K}_{a_i} \otimes \mathscr{H}_M$ in $\mathscr{H}_S \otimes \mathscr{H}_M$, do not belong to $\mathscr{D}_\sigma$. Nor do the elements in $\mathscr{L}$ corresponding to the indicator values of M belong to $\mathscr{D}_\sigma$, i.e., the elements corresponding to the subspaces $\mathscr{H}_S \otimes \mathscr{K}_{r_i}$ in $\mathscr{H}_S \otimes \mathscr{H}_M$. So neither the measured properties of S nor the indicator properties of M are determinate in the state σ.

What we appear to want at time t is not the state σ but the mixture $W = \sum |c_i|^2 P_{a_i} \otimes P_{r_i}$, because the transition $\psi \otimes \rho_0 \rightarrow W$ satisfies (a) and (b), as well as (c). But no unitary transformation can accomplish the transition from a pure state to a mixture, and so the measurement problem appears to be an inescapable feature of any theory in which a Boolean property structure $\mathscr{B}$ is replaced by a non-Boolean structure $\mathscr{L}$ that is not, in the general case, imbeddable into any $\mathscr{B}$.

This problem is often dramatized as the problem of Schrodinger's cat, where macroscopic properties of a cat become indeterminate in a measurement interaction. Suppose $\mathscr{H}$ is 2-dimensional and M is a cat, with ρ_1 representing a quantum state of a live cat and ρ_2 representing a quantum state of a dead cat. (We assume that the Hilbert space representing quantum states—microstates—of the cat can be decomposed into the span of two subspaces: $\mathscr{K}_{\text{alive}}$, the subspace of quantum states in which the cat is alive, and $\mathscr{K}_{\text{dead}}$, the subspace of quantum states in which the cat is dead. So $\rho_1 \in \mathscr{K}_{\text{alive}}$ and $\rho_2 \in \mathscr{K}_{\text{dead}}$.) Then, after the interaction with S, the cat is alive if the final state of the composite system is $\alpha_1 \otimes \rho_1$, the cat is dead if the final state is $\alpha_2 \otimes \rho_2$, but the cat is neither alive nor dead if the final state is the linear superposition $c_1 \alpha_1 \otimes \rho_1 + c_2 \alpha_2 \otimes \rho_2$. This is because neither the cat-property "alive" (represented by the subspace $\mathscr{K}_{\text{alive}}$) nor the cat-property "dead" (represented by the subspace $\mathscr{K}_{\text{dead}}$) belongs to $\mathscr{D}_\sigma$ if

$\sigma = c_1\alpha_1 \otimes \rho_1 + c_2\alpha_2 \otimes \rho_2$, and so both these cat-properties are indeterminate in this state.

But there is another constraint concerning cats that the theory fails to satisfy, a second cat problem, at least if we are prepared to accept the assumptions underlying the first problem. This second cat problem may be dramatized this way: Take a cat that is initially dead, i.e., initially represented by some quantum state in the subspace $\mathscr{H}_{\text{dead}}$ (and so determinately dead by the principle D). Measure a dynamical quantity Y that does not commute with the dynamical quantity represented by the operator $X = P_{\text{alive}} - P_{\text{dead}}$ (where P_{alive} is the projection operator onto the subspace $\mathscr{H}_{\text{alive}}$ and P_{dead} is the projection operator onto the subspace $\mathscr{H}_{\text{dead}}$). Note that X takes the value $+1$ if the cat is alive and -1 if the cat is dead. According to quantum mechanics, there will now be a finite probability that X takes the value $+1$, i.e., that the cat is alive, more precisely that the quantity X will be found to have changed its value from -1 to $+1$. Since we take it for granted that dead cats cannot be resurrected, the failure of the theory to satisfy this constraint is clearly also a problem. [I think I first heard this problem years ago from Constantin Piron. Bas van Fraassen formulates a related problem—he calls it the "benign cat paradox"—in his book (van Fraassen, 1991).]

The second cat problem exploits a pervasive feature of quantum mechanics: if we measure a sequence of dynamical quantities, X-Y-X, where X and Y do not commute, we will not necessarily obtain the same value for X on both measurements. In fact, this is a paradigm case of interference, the way in which noncommutativity manifests itself. If we deny the assumptions underlying the second cat problem (e.g., the appropriateness of a quantum mechanical description of the properties "alive" and "dead" of a cat, the existence of a dynamical quantity like Y that does not commute with X), we also deny the assumptions underlying Schrodinger's original cat problem. (Note that in the original problem we effectively assume the existence of a quantity P_σ, $\sigma = c_1\alpha_1 \otimes \rho_1 + c_2\alpha_2 \otimes \rho_2$ that does not commute with the quantity $A \otimes X$ of the composite systems S + cat.)

I suggest that the second cat problem—the resurrection problem—must be resolved before we can hope to find a solution to Schrodinger's original cat problem, or more generally, a solution to the measurement problem. It seems clear that we cannot avoid the resurrection problem without imposing a restriction on the superposition principle, i.e., without introducing superselection rules in the theory. If we want it to be impossible to resurrect dead cats in a quantum mechanical universe, then certain linear superpositions of cat-states must be theoretically impossible. More generally, if we want to model, in the formalism of quantum mechanics, systems (like macrosystems) that are characterized by *some* (not necessarily all) dynamical quantities that commute with all quantities, then there must be a restriction on the superposition principle for such "quasiclassical" systems that do not exhibit interference effects for all quantities. (Formally, such quasiclassical systems would be represented by algebras of dynamical quantities with nontrivial centers, i.e., algebras in which some quantities, aside from the unit, commute with all quantities.)

The introduction of (discrete) superselection rules partitions the Hilbert space of a system into a countable set of orthogonal subspaces $\mathscr{K}_j$, the superselection sectors or coherent subspaces of the system. A quasiclassical quantity R has the spectral representation $R = \sum r_j P_j$, where P_j is the projection operator onto the coherent subspace K_j. Other quantities are required to leave the coherent subspaces invariant, i.e., to map vectors in $\mathscr{K}_j$ into $\mathscr{K}_j$, for all j. So the projection operators in the spectral representation of self-adjoint operators representing dynamical quantities all project onto subspaces of the $\mathscr{K}_j$. Other self-adjoint operators do not represent dynamical quantities of the system. So, every dynamical quantity is represented by a self-adjoint operator on the Hilbert space $\mathscr{H}$ of the system, but not every self-adjoint operator on $\mathscr{H}$ represents a dynamical quantity.

Similarly, every pure state is represented by a unit vector in $\mathscr{H}$, but not every unit vector in $\mathscr{H}$ represents a pure state. Vectors represented by linear superpositions of vectors belonging to different coherent subspaces $\mathscr{K}_j$ represent mixtures. If $\phi = \sum d_j \rho_j$, $\rho_j \in \mathscr{K}_j$, then, for every projection operator P onto a subspace of some coherent subspace $\mathscr{K}_j$, we have

$$(\phi, P\phi) = \sum |d_j|^2 (\rho_j, P\rho_j) = \mathrm{Tr}(WP)$$

where $W = \sum |d_j|^2 P_{\rho_j}$, and so ϕ and W assign the same probabilities to all ranges of values of all dynamical quantities. In a modified quantum mechanics with superselection rules, there is no longer a 1–1 correspondence between states and statistical operators (density matrices) on $\mathscr{H}$—there are no longer sufficient dynamical quantities to separate the set of statistical operators on H.

The problem of modeling quasiclassical systems in quantum mechanics is a nontrivial problem that requires a solution, independently of any solution to the problem of modeling measurement processes in the theory. Once this problem is resolved—and a solution surely requires introducing some restriction on the principle of superposition for macroscopic systems with a large number of degrees of freedom, such as cats and measuring instruments (or perhaps deriving such a restriction for systems with infinitely many degrees of freedom)—there is no further formal obstacle to a solution of the measurement problem. The effect of a restriction on the superposition principle is to guarantee the existence of quasiclassical systems with fixed

Boolean algebras of determinate properties, i.e., Boolean algebras of properties that are always determinate, irrespective of the quantum state of the system. A macroscopic measuring instrument with n "pointer readings" or indicator values would then be modeled as such a system, and associated with a fixed Boolean algebra of determinate properties generated by n atoms corresponding to the properties represented by the indicator values. Similarly, a cat would be modeled as such a system, with the dynamical quantity X in the center of the algebra associated with a fixed Boolean algebra of determinate properties (so the cat would always be either determinately alive or, exclusively, determinately dead).

The measurement problem then becomes the problem of showing that, given the initial quantum state ψ of a system S and the initial quantum state ρ_0 of a second system M (the measuring instrument) at some time t_0, where ρ_0 represents the zero indicator state of M, *and the indicator values are values of a quasiclassical dynamical quantity R of M and so correspond to the atoms of a fixed Boolean subalgebra $\mathscr{B}$ of determinate properties in $\mathscr{L}$*, there exists an interaction that transforms the state of the composite system $\psi \otimes \rho_0$ to a new state $\sigma \in \mathscr{H}_S \otimes \mathscr{H}_M$ at time t, in which the properties of S associated with the measured quantity A of S are (a) determinate in the state σ, and (b) correlated with the indicator values of M in the state σ.

3. A MEASUREMENT THEOREM

The possibility that a quantum system might be characterized by a fixed Boolean algebra $\mathscr{B}$ of determinate properties requires a modification of the principle D. As formulated, the principle of determinateness says that a certain set of properties $\mathscr{D}_\sigma$ is determinate for a composite system $S+M$ in the state σ. Suppose M is associated with a fixed Boolean algebra of determinate properties $\mathscr{B}_M$. Then D will in general select a different set of determinate properties for M, via $\mathscr{D}_\sigma$, than the actual fixed set $\mathscr{B}_M$.

What we want is to modify the principle D so that if a system is associated with a fixed Boolean algebra $\mathscr{B}$ of determinate properties, then we take as determinate (in the state ψ) the union of all the Boolean algebras defined by completion from the elements in the set $D_\psi = \bigcup \mathscr{B}_\psi$, given by the principle D, that are compatible with the elements in $\mathscr{B}$, together with $\mathscr{B}$. That is, for each $\mathscr{B}_\psi$, we take the elements in $\mathscr{B}_\psi$ that are compatible with $\mathscr{B}$, together with the elements in $\mathscr{B}$, and complete the Boolean algebra by taking meets (infima), joins (suprema), and complements. Then we take the union over all these Boolean algebras as the set of properties that are determinate for the system in the state ψ.

Note that completing the algebras in this way introduces new elements that do not belong to $\mathscr{D}_\psi$ or to $\mathscr{B}$. I shall write this set as $\mathscr{D}_\psi/\mathscr{B}$, where $\mathscr{Y}/\mathscr{X}$

indicates the operation of forming the union of the extensions of $\mathscr{X}$ to all Boolean algebras generated from $\mathscr{X}$ via meets, joins, and complements by the elements of $\mathscr{Y}$ that are compatible with $\mathscr{X}$. As the "/" suggests, the operation is meant to be understood as a kind of conditionalization, or minimal revision of $\mathscr{Y}$ by $\mathscr{X}$.

The following "modified principle of determinateness" captures this notion:

D*: In a non-Boolean property structure $\mathscr{L}$ associated with a quantum mechanical system S with a fixed Boolean subalgebra $\mathscr{B}$ of determinate properties, the elements that are determinate in the state ψ are those elements e that belong to the extension of $\mathscr{B}$ to any Boolean subalgebra of $\mathscr{L}$ generated by completion from all the elements in $\mathscr{B}$, together with those elements in $\mathscr{D}_\psi = \bigcup \mathscr{B}_\psi$ (defined according to the original principle D) that are compatible with the elements in $\mathscr{B}$. This set of elements is denoted by $\mathscr{D}_\psi/\mathscr{B}$. (By "compatible" here I mean that one of the corresponding subspaces is contained in the other, or the two subspaces are orthogonal except for an overlap, where the overlap is the infimum or intersection of the subspaces. Equivalently, the corresponding projection operators commute.)

A measurement theorem can now be proved for the principle D*:

Theorem. D* ensures that the Boolean algebra of properties associated with (ranges of) values of an indeterminate dynamical quantity A of a system S becomes determinate in an appropriate (measurement) interaction with a quasiclassical system M, i.e., a system with a fixed determinate Boolean algebra of properties associated with a quasiclassical quantity R, and no other incompatible Boolean algebra of properties in the property structure of S becomes determinate in the interaction. Moreover, no quantities Q of the composite system $S + M$ that are not of the form $X \otimes Y$ become determinate in the interaction, and no quantity of the quasiclassical system M is determinate after the interaction except R and refinements of R.

Proof. Suppose we measure a quantity A of S, with eigenvalues $a_1, \ldots, a_n$ and corresponding eigenvectors $\alpha_1, \ldots, \alpha_n$, by an interaction with an instrument M with a fixed Boolean algebra of determinate properties B_M, generated by the $n+1$ atoms corresponding to the projection operators $P_0, P_1, \ldots, P_n$ onto $n+1$ multidimensional (perhaps ∞-dimensional) subspaces $\mathscr{K}_i$ of $\mathscr{H}_M$. The atoms of $\mathscr{B}_M$ represent the indicator values of M. Suppose the initial state of S is $\psi = \sum c_i \alpha_i$ and the initial state of M is $\rho_0 \in \mathscr{K}_0$. By the linearity assumption, the measurement interaction induces the transition

$$\psi \otimes \rho_0 \rightarrow \sigma = \sum c_i \alpha_i \otimes \rho_i$$

By D*, after the measurement interaction, when the state of $S+M$ is σ, the determinate properties of $S+M$ are given by $\bigcup \mathscr{B}_\sigma/\mathscr{B}_M$. What properties of $S+M$ belong to $\bigcup \mathscr{B}_\sigma/\mathscr{B}_M$?

To answer this question, first consider the subspace $\mathscr{H}_S\otimes\mathscr{H}'_M$ of $\mathscr{H}\otimes\mathscr{H}_M$, where $\mathscr{H}'_M$ is spanned by the vectors $\rho_1, \ldots, \rho_n$. Note that $\sigma\in\mathscr{H}_S\otimes\mathscr{H}'_M$. Now construct $\bigcup \mathscr{B}'_\sigma/\mathscr{B}'_M$ (the restriction of $\bigcup \mathscr{B}_\sigma/\mathscr{B}_M$ to $\mathscr{H}_S\otimes\mathscr{H}'_M$), where $\mathscr{B}'_\sigma$ is a maximal Boolean subalgebra in the restricted property structure $\mathscr{L}'$ of $\mathscr{H}_S\otimes\mathscr{H}'_M$, generated by the atoms corresponding to σ and any n^2-1 vectors orthogonal to σ in $\mathscr{H}_S\otimes\mathscr{H}'_M$, and $\mathscr{B}'_M$ is the Boolean subalgebra generated by the atoms corresponding to the projection operators $I\otimes P_{\rho_1}, \ldots, I\otimes P_{\rho_n}$.

Now, among the $\mathscr{B}'_\sigma$ there is a maximal Boolean subalgebra $\mathscr{B}''_\sigma$ generated by the atoms corresponding to the n^2-n orthonormal vectors $\alpha_i\otimes\rho_j$, $i\neq j$ (which are all orthogonal to $\sigma=\sum c_i\alpha_i\otimes\rho_i$), and any n orthonormal vectors $\sigma, \sigma_2, \ldots, \sigma_n(\neq\alpha_i\otimes\rho_j)$ spanning the n-dimensional subspace $\mathscr{K}\in\mathscr{H}_S\otimes\mathscr{H}'_M$. Since $\mathscr{K}$ is orthogonal to the subspace spanned by the vectors $\alpha_i\otimes\rho_j$ $(i\neq j)$, it follows that $\mathscr{K}$ is spanned by the vectors $\alpha_i\otimes\rho_i$, $i=1, \ldots, n$. Note that the set $\{\alpha_i\otimes\rho_j(i\neq j); \sigma, \sigma_2, \ldots, \sigma_n\}$ forms an orthonormal basis in $\mathscr{H}_S\otimes\mathscr{H}'_M$. The remaining $\mathscr{B}'_\sigma$ in the union are generated from atoms corresponding to σ and other sets of orthonormal basis vectors in $\mathscr{K}^{\#}_\sigma=(\mathscr{H}_S\otimes\mathscr{H}'_M)-\mathscr{K}_\sigma$, the orthogonal complement of $\mathscr{K}_\sigma$ in $\mathscr{H}_S\otimes\mathscr{H}'_M$.

All n^2-n atoms in $\mathscr{B}''_\sigma$ corresponding to the vectors $\alpha_i\otimes\rho_j$ $(i\neq j)$ are compatible with $\mathscr{B}'_M$, and none of the atoms corresponding to the vectors $\sigma, \sigma_2, \ldots, \sigma_n$ are compatible with $\mathscr{B}'_M$. Let P be the projection operator onto the n-dimensional subspace $\mathscr{K}$, i.e.,

$$P=P_\sigma+P_{\sigma_2}+\cdots+P_{\sigma_n}=P_{\alpha_1}\otimes P_{\rho_1}+\cdots+P_{\alpha_n}\otimes P_{\rho_n}$$

Clearly the subspaces corresponding to P and $I\otimes P_{\rho_k}$ are orthogonal except for an overlap (the 1-dimensional subspace spanned by $\alpha_k\otimes\rho_k$, for each k), and so the corresponding elements are compatible. Equivalently, P commutes with $I\otimes P_{\rho_k}$ $(k=1, \ldots, n)$, and the infimum of the elements corresponding to P and $I\otimes P_{\rho_k}$ is $P_{\alpha_k}\otimes P_{\rho_k}$, for $k=1, \ldots, n$:

$$P\wedge I\otimes P_{\rho_k}=P\cdot I\otimes P_{\rho_k}=P_{\alpha_k}\otimes P_{\rho_k}$$

So $\mathscr{B}''_\sigma/\mathscr{B}'_M$ is just the maximal Boolean subalgebra $\mathscr{B}'_{AM}$ in $\mathscr{L}'$ generated by the n^2 atoms corresponding to the vectors $\alpha_i\otimes\rho_j$ $(i=1, \ldots, n; j=1, \ldots, n)$.

Extending $\mathscr{H}'_M$ to $\mathscr{H}_M$ (hence $\mathscr{B}'_\sigma$ to $\mathscr{B}_\sigma$, $\mathscr{B}'_M$ to $\mathscr{B}_M$, and $\mathscr{L}'$ to $\mathscr{L}$) involves adding the subspaces $\mathscr{K}_0, \mathscr{K}_i-\mathscr{K}_{\rho_i}$, $i=1, \ldots, n$. Here $\mathscr{B}_M$ is the nonmaximal Boolean subalgebra of $\mathscr{L}$ generated by the $n+1$ atoms corresponding to the projection operators $I\otimes P_0, \ldots, I\otimes P_n$. The Boolean algebra $\mathscr{B}'_{AM}$ is nonmaximal in $\mathscr{L}$ and is contained in a family of maximal

Boolean subalgebras of $\mathscr{L}$ that differ from $\mathscr{B}'_{AM}$ by the addition of atoms corresponding to the tensor products of the vectors α_i with different choices of orthogonal vectors $\rho_{i,u}$ $[u=1,\ldots,m(i)]$ spanning the subspace $\mathscr{K}_i-\mathscr{K}_{\rho_i}$, for $i=1,\ldots,n$, and different choices of orthogonal vectors spanning the subspace $\mathscr{K}_0$ (assuming, for simplicity, that $\mathscr{H}_M$ is finite-dimensional, i.e., that each coherent subspace $\mathscr{K}_i$, $i=1,\ldots,n$, is $m(i)$-dimensional). Each of these algebras contains the subalgebra $\mathscr{B}_{AM}$ generated by the $n(n+1)$ atoms corresponding to the projection operators $P_{\alpha_i}\otimes P_j$ $(i=1,\ldots,n; j=0,\ldots,n)$, i.e., the nonmaximal Boolean subalgebra of properties corresponding to the indicator values of M and the properties of S measured by M. (Note that the subalgebra generated by the n^2 atoms corresponding to the projection operators $P_{\alpha_i}\otimes P_j$ $(i=1,\ldots,n; =1,\ldots,n)$ is isomorphic to $\mathscr{B}'_{AM}$.)

What has been demonstrated is that $\bigcup \mathscr{B}_\sigma/\mathscr{B}_M$ contains $\mathscr{B}_{AM}$, together with refinements of $\mathscr{B}_{AM}$ generated by adding elements corresponding to subspaces contained in the subspaces $\mathscr{K}_{\alpha_i}\otimes\mathscr{K}_i$ (elements corresponding to refinements of the properties represented by the values of the indicator quantities of M). So, after the measurement, it follows from D* that the properties of S measured by M are determinate, assuming that the Boolean algebra of indicator values of M is a fixed determinate set of properties of M.

Moreover, there are no other properties of S, incompatible with the measured A-properties, that become determinate in the measurement interaction. For suppose there is some quantity B incompatible (i.e., noncommuting) with A that becomes determinate in the measurement interaction, as well as A. There are two possibilities to consider here: Either the eigenvectors of B are related to the eigenvectors of A by a linear transformation involving all the eigenvectors α_i in $\mathscr{H}_S$, or one or more of the eigenvectors α_i remain fixed in the transformation.

Again, consider initially the subspace $\mathscr{H}_S\otimes\mathscr{H}'_M$. In the first case, no vectors of the form $\beta_i\otimes\rho_j$ can be derived from a linear transformation of the vectors $\{\alpha_i\otimes\rho_j(i\neq j);\ \sigma_2,\ldots,\sigma_n\}$ orthogonal to σ in the subspace $\mathscr{K}^{\#}_\sigma$. To see this, first note that the vectors $\alpha_j\otimes\rho_j$ $(j=1,\ldots,n)$ lie in the n-dimensional subspace $\mathscr{K}$ and not in the subspace spanned by $\sigma_2,\ldots,\sigma_n$ without σ, unless some of the coefficients in σ are zero. Now $\beta_i=\sum b_{ij}\alpha_j$, and by assumption $b_{ij}\neq 0$ $(j=1,\ldots,n)$ so the coefficients of the terms $\alpha_j\otimes\rho_j$ $(j=1,\ldots,n)$ in the representation of the $\beta_i\otimes\rho_j$ in the basis $\alpha_i\otimes\rho_j$ $(i=1,\ldots,n; j=1,\ldots,n)$ will be nonzero, which means that no vector of the form $\beta_i\otimes\rho_j$ lies in the subspace $\mathscr{K}^{\#}_\sigma$, for any i, j. It follows that if the Boolean subalgebra $\mathscr{B}'_\sigma/\mathscr{B}'_M\supseteq\mathscr{B}'_{BM}$, for some B incompatible with A, then all the n^2 atoms of $\mathscr{B}'_{BM}$ must be generated as the infima of the elements corresponding to $I\otimes P_{\rho_k}$ $(k=1,\ldots,n)$ with elements corresponding to subspaces spanned by vectors obtained as linear superpositions of the vectors

$\{\alpha_i \otimes \rho_j (i \neq j);\ \sigma_2, \ldots, \sigma_n\}$. But at most n of the required n^2 atoms can be generated this way.

In the second case, if we fix one of the eigenvectors, say $\alpha_1 = \beta_1$, then at most $2(n-1) = 2n-2$ vectors of the form $\beta_i \otimes \rho_j$ can be derived from a linear transformation of the vectors $\{\alpha_i \otimes \rho_j (i \neq j);\ \sigma_2, \ldots, \sigma_n\}$ in $\mathcal{K}^{\#}_{\sigma}$. (Note that $\beta_1 \otimes \rho_1 = \alpha_1 \otimes \rho_1$ cannot be derived in this way because it does not lie in the subspace $\mathcal{K}^{\#}_{\sigma}$.) With at most n additional atoms generated via infima from the elements corresponding to $I \otimes P_{\rho_k}$ $(k = 1, \ldots, n)$, this yields at total of at most $3n-2$ atoms, which is less than the n^2 atoms required to generate $\mathscr{B}'_{BM}$, for all $n > 1$.

To see this, consider first a 3-dimensional Hilbert space for $\mathscr{H}_S$ (i.e., $n = 3$). Suppose $\beta_1 = \alpha_1$, $\beta_2 = b_{22}\alpha_2 + b_{23}\alpha_3$, and $\beta_3 = b_{32}\alpha_2 + b_{33}\alpha_3$. The $n-1 = 2$ vectors $\beta_1 \otimes \rho_j = \alpha_1 \otimes \rho_j$ $(j = 2, 3)$ and the $n-1 = 2$ vectors $\beta_2 \otimes \rho_1 = (b_{22}\alpha_2 + b_{23}\alpha_3) \otimes \rho_1$ and $\beta_3 \otimes \rho_1 = (b_{32}\alpha_2 + b_{33}\alpha_3) \otimes \rho_1$ all belong to $\mathcal{K}^{\#}_{\sigma}$. But the 5 vectors $\beta_1 \otimes \rho_1$, $\beta_2 \otimes \rho_2$, $\beta_2 \otimes \rho_3$, $\beta_3 \otimes \rho_2$, and $\beta_3 \otimes \rho_3$ do not belong to $\mathcal{K}^{\#}_{\sigma}$, because the coefficients of a term of the form $\alpha_i \otimes \rho_j$, for some j, will be nonzero in the representation of each of these vectors in the basis $\alpha_i \otimes \rho_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, n)$. (For example, $\beta_2 \otimes \rho_2 = (b_{22}\alpha_2 + b_{23}\alpha_3) \otimes \rho_2$ and $b_{22} \neq 0$.) At most 3 of the 5 atoms corresponding to these vectors can be generated by infima from the elements corresponding to $I \otimes P_{\rho_k}$ $(k = 1, 2, 3)$.

If $\mathscr{H}_S$ has more than 3 dimensions, we can keep one or more vectors fixed in the transformation from the A-basis to the B-basis, but in each case fewer than the required number of atoms for $\mathscr{B}'_{BM}$ can be obtained by transformation of the vectors $(\alpha_i \otimes \rho_j (i \neq j);\ \sigma_2, \ldots, \sigma_n\}$ in $\mathcal{K}^{\#}_{\sigma}$ together with infima generated via the $I \otimes P_{\rho_k}$ $(k = 1, \ldots, n)$. If we keep one vector fixed, we again obtain at most $2(n-1) = 2n-2$ vectors by transformation of the vectors $\{\alpha_i \otimes \rho_j (i \neq j);\ \sigma_2, \ldots, \sigma_n\}$ in $\mathcal{K}^{\#}_{\sigma}$. If we keep two vectors fixed, at most $2(n-1) + 2(n-2) = 4n-6$ vectors can be obtained by transformation in the subspace $\mathcal{K}^{\#}_{\sigma}$. If we keep three vectors fixed, at most $2(n-1) + 2(n-2) + 2(n-3) = 6n-12$ vectors can be obtained by transformation in $\mathcal{K}^{\#}_{\sigma}$. In each case, adding at most n elements generated via infima from the $I \otimes P_{\rho_k}$, we get too few atoms to generate $\mathscr{B}'_{BM}$, i.e., $3n-2 < n^2$ for $n \geq 3$; $5n-6 < n^2$ for $n \geq 4$; $7n-12 < n^2$ for $n \geq 5$; and so on.

There are clearly other formal quantities of $S+M$ (represented by self-adjoint operators in $\mathscr{H}_S \otimes \mathscr{H}_M$) that become determinate in the measurement interaction on the basis of the principle D* alone. For example, suppose we transform $\mathscr{B}''_{\sigma}$ to $\mathscr{B}'''_{\sigma}$ by transforming $\alpha_2 \otimes \rho_1$ and $\alpha_3 \otimes \rho_1$ to

$$\beta_1 = (1/\sqrt{2})\alpha_2 \otimes \rho_1 + (1/\sqrt{2})\alpha_3 \otimes \rho_1 = (1/\sqrt{2})(\alpha_2 + \alpha_3) \otimes \rho_1$$

$$\beta_2 = (1/\sqrt{2})\alpha_2 \otimes \rho_1 - (1\sqrt{2})\alpha_3 \otimes \rho_1 = (1/\sqrt{2})(\alpha_2 - \alpha_3) \otimes \rho_1$$

respectively. Then the atoms corresponding to β_1 and β_2 are compatible with the elements corresponding to $I \otimes P_{\rho_k}$ $(k=0, 1, \ldots, n)$ and $\mathscr{B}'''_{\sigma}/\mathscr{B}'_M = \mathscr{B}' \neq \mathscr{B}'_{AM}$. Here $\mathscr{B}'$ is a maximal Boolean subalgebra of properties in $\mathscr{L}'$, assuming $\mathscr{L}'$ is the full property structure derived from $\mathscr{H}_S \otimes \mathscr{H}'_M$ without superselection rules. Extending $\mathscr{H}_S \otimes \mathscr{H}'_M$ to $\mathscr{H}_S \otimes \mathscr{H}_M$ and $\mathscr{L}'$ to $\mathscr{L}$ yields a nonmaximal Boolean subalgebra of properties corresponding to some quantity Q of the composite system, with eigenvectors including the set $\{\alpha_i \otimes \rho_j | i=1, \ldots, n; j=1, \ldots, n; \alpha_2 \otimes \rho_1$ replaced by β_1; $\alpha_3 \otimes \rho_1$ replaced by $\beta_2\}$. [The eigenspaces represented by the projection operators $P_{\alpha_i} \otimes P_0$ and $P_{\alpha_i} \otimes (P_j - P_{\rho_j})$, $i=1, \ldots, n; j=1, \ldots, n$, are also eigenspaces of Q.] So Q cannot be a dynamical quantity of the form $X \otimes Y$.

Now none of these quantities, formally determinate on the basis of D*, can represent dynamical quantities of $S+M$. If M is characterized by superselection rules, the effect is to restrict the set of quantities that represent dynamical quantities to those operators representable as linear sums of projection operators onto the coherent subspaces of $\mathscr{H}_S \otimes \mathscr{H}_M$. Now, $S+M$ will also be subject to superselection rules and so quantities like Q, as well as the property (idempotent quantity) associated with P_s, will no longer be dynamical quantities of $S+M$.

This measurement theorem establishes that we can consistently interpret the non-Boolean algebra of idempotent dynamical quantities of a quantum mechanical system as a property structure analogous to the Boolean property structure of a classical mechanical system. The dynamical quantities of a quantum mechanical system are not all simultaneously determinate, i.e., they do not all possess values simultaneously. The set of dynamical quantities that are determinate for a system S is specified by the quantum state according to the principle D*. Indeterminate quantities of S become determinate in suitable interactions with quasiclassical systems, and these interactions can be interpreted as measurements.

I do not suggest here that the notion of a property that can sometimes be "determinate" and sometimes "indeterminate" is clear in a metaphysical sense, on the basis of the above analysis. The point of the analysis is to show that, once we grant that the determinate–indeterminate dichotomy applies to properties, the non-Boolean algebra of idempotent quantities of a quantum mechanical system can be interpreted as a property structure in which the transition from indeterminateness to determinateness arises as the result of interactions involving quasiclassical systems.

What we require, in order to interpret the non-Boolean algebra of idempotent dynamical quantities of a quantum mechanical system as a property structure, is *a principle that describes how determinateness is transformed*, just as we require a principle that describes how truth is transformed in the Boolean property structures of classical mechanics, in which all properties are simultaneously determinate. And this is provided by the principle D*. The equations of motion of classical mechanics do not generate values for dynamical quantities, or truth values for propositions. Rather, *given* initial values for certain quantities, or the truth values of certain propositions, classical mechanics shows how these values are *transformed* at other times (an initial assignment of values to certain dynamical quantities, or constraints on the values of these quantities, determines values or constraints on values to these or other quantities at other times). Similarly, the unitary transformations of quantum mechanics cannot be expected to generate determinateness. Rather, *given* that certain quantities are determinate initially, the theory should show how determinateness is *transformed* and what quantities are determinate at later times. And this is accomplished, according to the principle D*, once we are guaranteed the existence of quasiclassical systems with fixed determinate sets of properties in quantum mechanics.

REFERENCE

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